

積分の計算《基本演習》 (NO.1) 解答例 1 枚目

1. 次の不定積分を求めよ。

(1) $\int \frac{x+1}{x^2} dx$
 (解) $\int \frac{x+1}{x^2} dx = \int \left(\frac{1}{x} + \frac{1}{x^2} \right) dx$
 $= \int \left(\frac{1}{x} + x^{-2} \right) dx = \log|x| + \frac{1}{-2+1} x^{-2+1} + c$
 $= \log|x| - x^{-1} + c = \log|x| - \frac{1}{x} + c$ "

(2) $\int e^{3-2x} dx$
 (解) $3-2x=t$ とおくと、 $\frac{dt}{dx} = -2$ $-\frac{1}{2}dt = dx$
 $\int e^{3-2x} dx = -\frac{1}{2} \int e^t dt = -\frac{1}{2}e^t + c$
 $= -\frac{1}{2}e^{3-2x} + c$ "
 (別解) $\int e^x dx = e^x$ だから
 $\int e^{3-2x} dx = \frac{1}{-2}e^{3-2x} + c = -\frac{1}{2}e^{3-2x} + c$ "

(3) $\int (3x+2)^4 dx$
 (解) $3x+2=u$ とおくと、 $\frac{du}{dx} = 3$ $\frac{1}{3}du = dx$
 $\int (3x+2)^4 dx = \int u^4 \cdot \frac{1}{3} du = \frac{1}{3} \int u^4 du$
 $= \frac{1}{3} \cdot \frac{1}{5} u^5 + c = \frac{1}{15} (3x+2)^5 + c$ "
 (別解) $\int x^4 dx = \frac{1}{4+1} x^{4+1} = \frac{1}{5} x^5$ だから
 $\int (3x+2)^4 dx$
 $= \frac{1}{3} \cdot \frac{1}{5} (3x+2)^5 + c = \frac{1}{15} (3x+2)^5 + c$ "

(4) $\int \tan x dx$
 (解) $\int \tan x dx = \int \frac{\sin x}{\cos x} = \int \left(-\frac{-\sin x}{\cos x} \right) dx$
 $= - \int \frac{(\cos x)}{\cos x} dx = -\log|\cos x| + c$ "
 (別解) $\cos x = t$ とおくと、
 $\frac{dt}{dx} = -\sin x$ $\sin x dx = -dt$
 $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} \cdot \sin x dx$
 $= \int \frac{1}{t} (-1) dt = - \int \frac{1}{t} dt$
 $= -\log|t| + c = -\log|\cos x| + c$ "

(5) $\int x \cos x dx$
 (解) $\int \cos x dx = \sin x$ より、部分積分法で解くと
 $\int x \cos x dx = x \sin x - \int (x) \sin x dx$
 $= x \sin x - \int \sin x dx = x \sin x + \cos x + c$ "

(6) $\int \frac{x}{\sqrt{1+x}} dx$
 (解) $1+x=t$ とおくと、 $\frac{dx}{dt} = 1$ $dx = dt$
 $\int \frac{x}{\sqrt{1+x}} dx = \int \frac{t-1}{\sqrt{t}} dt = \int \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right) dt$
 $= \int (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) dt = \frac{1}{\frac{3}{2}} t^{\frac{3}{2}} - \frac{1}{\frac{1}{2}} t^{\frac{1}{2}} + c = \frac{2}{3} t\sqrt{t} - 2\sqrt{t} + c$
 $= \frac{2}{3} \sqrt{t} (t-3) + c = \frac{2}{3} (x-2)\sqrt{1+x} + c$ "

(7) $\int \frac{x}{\sqrt{1+x^2}} dx$
 (解) $1+x^2=t$ とおくと、 $\frac{dt}{dx} = 2x$ $\frac{1}{2}dt = x dx$
 $\int \frac{x}{\sqrt{1+x^2}} dx = \int \frac{1}{\sqrt{t}} \cdot \frac{1}{2} dt = \frac{1}{2} \int t^{-\frac{1}{2}} dt$
 $= \frac{1}{2} \cdot \frac{1}{-\frac{1}{2}+1} t^{-\frac{1}{2}+1} + C = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} t^{\frac{1}{2}} = \sqrt{t} + C$
 $= \sqrt{1+x^2} + C$ "

(8) $\int x^2 e^{4x} dx$
 (解) $\int e^{4x} dx = \frac{1}{4} e^{4x}$ より、部分積分法で解くと
 $\int x^2 e^{4x} dx = x^2 \left(\frac{1}{4} e^{4x} \right) - \int (x^2) \cdot \left(\frac{1}{4} e^{4x} \right) dx$
 $= \frac{1}{4} x^2 e^{4x} - \int 2x \cdot \frac{1}{4} e^{4x} dx = \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \int x e^{4x} dx$
 $= \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \left\{ x \cdot \left(\frac{1}{4} e^{4x} \right) - \int x \cdot \left(\frac{1}{4} e^{4x} \right) dx \right\}$
 $= \frac{1}{4} x^2 e^{4x} - \frac{1}{8} x e^{4x} + \frac{1}{8} \int e^{4x} dx$
 $= \frac{1}{4} x^2 e^{4x} - \frac{1}{8} x e^{4x} + \frac{1}{32} e^{4x} + C$
 $= \frac{1}{32} e^{4x} (8x^2 - 4x + 1) + C$ "

(9) $\int \cos^3 x dx$
 (解) 与式 $= \int \cos^2 x \cos x dx = \int (1 - \sin^2 x) \cos x dx$
 $\sin x = t$ とおくと、 $\frac{dt}{dx} = \cos x$ $dt = \cos x dx$
 与式 $= \int (1-t^2) dt = t - \frac{1}{3} t^3 + C$
 $= \sin x - \frac{1}{3} \sin^3 x + C$ "

積分の計算《基本演習》 (NO.1) 解答例 2 枚目

2. 次の定積分の値を求めよ。

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin x + \cos x) dx$

(解) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin x + \cos x) dx = [-\cos x + \sin x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$
 $= (-\cos \frac{\pi}{3} + \sin \frac{\pi}{3}) - (-\cos \frac{\pi}{6} + \sin \frac{\pi}{6})$
 $= (-\frac{1}{2} + \frac{\sqrt{3}}{2}) - (-\frac{\sqrt{3}}{2} + \frac{1}{2}) = \sqrt{3} - 1$ "

(2) $\int_1^2 \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx$

(解) 与式 $= \int_1^2 \left(x - 2 + \frac{1}{x} \right) dx = \left[\frac{1}{2}x^2 - 2x + \log|x| \right]_1^2$
 $= (2 - 4 + \log 2) - (\frac{1}{2} - 2 + \log 1) = \log 2 - \frac{1}{2}$ "

(3) $\int_0^4 \sqrt{2x+1} dx$

x	$0 \rightarrow 4$
t	$1 \rightarrow 9$

(解) $2x+1=t$ とおくと、
 $\frac{dt}{dx} = 2 \quad dx = \frac{1}{2} dt$
 $\int_0^4 \sqrt{2x+1} dx = \int_1^9 \sqrt{t} \cdot \frac{1}{2} dt = \frac{1}{2} \int_1^9 t^{\frac{1}{2}} dt$
 $= \frac{1}{2} \left[\frac{2}{3} t^{\frac{3}{2}} \right]_1^9 = \frac{1}{3} [t\sqrt{t}]_1^9 = \frac{1}{3} (9\sqrt{9} - 1\sqrt{1}) = \frac{26}{3}$ "

(4) $\int_0^1 x e^{-x} dx$

(解) $\int_0^1 e^{-x} dx = \frac{1}{-1} e^{-x} = -e^{-x}$ だから部分積分法で
 $\int_0^1 x e^{-x} dx = [x(-e^{-x})]_0^1 - \int_0^1 (x) (-e^{-x}) dx$
 $= [-x e^{-x}]_0^1 + \int_0^1 e^{-x} dx$
 $= [-x e^{-x}]_0^1 + [-e^{-x}]_0^1$
 $= \{(-1 \cdot e^{-1}) - (-0 \cdot e^0)\} + \{(-e^{-1}) - (-e^0)\}$
 $= -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e}$ "

(5) $\int_0^{\frac{\pi}{3}} \sin 2x dx$

(解) $2x=t$ とおくと

x	$0 \rightarrow \frac{\pi}{6}$
t	$0 \rightarrow \frac{\pi}{3}$

 $\frac{dt}{dx} = 2 \quad \frac{1}{2} dt = dx$
 $\int_0^{\frac{\pi}{3}} \sin 2x dx = \int_0^{\frac{\pi}{3}} \sin t \cdot \frac{1}{2} dt = \frac{1}{2} \int_0^{\frac{\pi}{3}} \sin t dt$
 $= \frac{1}{2} [-\cos t]_0^{\frac{\pi}{3}} = -\frac{1}{2} [\cos t]_0^{\frac{\pi}{3}}$
 $= -\frac{1}{2} \left(\cos \frac{\pi}{3} - \cos 0 \right) = -\frac{1}{2} \left(\frac{1}{2} - 1 \right) = \frac{1}{4}$ "

(別解) 与式 $= \left[\frac{1}{2} \cdot (-\cos 2x) \right]_0^{\frac{\pi}{3}} = -\frac{1}{2} [\cos 2x]_0^{\frac{\pi}{3}}$
 $= -\frac{1}{2} \left(\cos \frac{2\pi}{3} - \cos 0 \right) = -\frac{1}{2} \left(-\frac{1}{2} - 1 \right) = \frac{1}{4}$ "

(6) $\int_1^3 \log x dx$

(解) $\int 1 dx = x$ だから部分積分法で解くと

$\int_1^3 \log x dx = \int_1^3 1 \cdot \log x dx$
 $= [x \log x]_1^3 - \int_1^3 x (\log x) dx$
 $= [x \log x]_1^3 - \int_1^3 x \cdot \frac{1}{x} dx$
 $= [x \log x]_1^3 - \int_1^3 1 dx = [x \log x]_1^3 - [x]_1^3$
 $= (3 \log 3 - 1 \log 1) - (3 - 1) = 3 \log 3 - 2$ "

(7) $\int_{-1}^1 \sqrt{4-x^2} dx$ ($f(x)=\sqrt{4-x^2}$ は偶関数である。)

(解) $\int_{-1}^1 \sqrt{4-x^2} dx = 2 \int_0^1 \sqrt{2^2-x^2} dx$
 $= 2 \left[\frac{1}{2} \left\{ x \sqrt{2^2-x^2} + 2^2 \text{Sin}^{-1} \frac{x}{2} \right\} \right]_0^1$
 $= \left[x \sqrt{2^2-x^2} + 2^2 \text{Sin}^{-1} \frac{x}{2} \right]_0^1$
 $= \left(\sqrt{3} + 4 \text{Sin}^{-1} \frac{1}{2} \right) - (0 - 4 \text{Sin}^{-1} 0)$
 $= \sqrt{3} + 4 \cdot \frac{\pi}{6} + 4 \cdot 0 = \sqrt{3} + \frac{2}{3} \pi$ "

(別解)

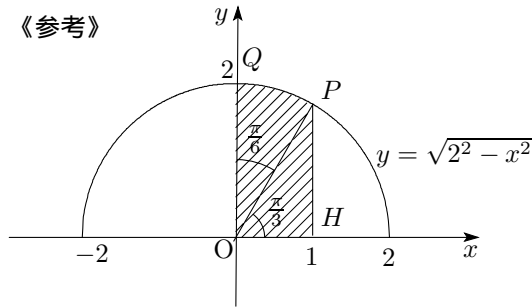
与式 $= 2 \int_0^1 \sqrt{2^2-x^2} dx$
 $x = 2 \sin \theta$ ($0 \leq \theta \leq \frac{\pi}{2}$) とおくと
 $\frac{dx}{d\theta} = 2 \cos \theta \quad dx = 2 \cos \theta d\theta$

x	$0 \rightarrow 1$
θ	$0 \rightarrow \frac{\pi}{6}$

 $\sqrt{2^2-x^2} = \sqrt{4-4 \sin^2 \theta} = 2\sqrt{1-\sin^2 \theta}$
 $= 2|\cos \theta| = 2 \cos \theta$

与式 $= 2 \int_0^{\frac{\pi}{6}} 2 \cos \theta \cdot 2 \cos \theta d\theta$
 $= 8 \int_0^{\frac{\pi}{6}} \cos^2 \theta d\theta = 8 \int_0^{\frac{\pi}{6}} \frac{1 + \cos 2\theta}{2} d\theta$
 $= 4 \int_0^{\frac{\pi}{6}} (1 + \cos 2\theta) d\theta = 4 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}}$
 $= 4 \left\{ \left(\frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} - (0 - 0) \right) \right\}$
 $= 4 \left(\frac{\pi}{6} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) = \frac{2}{3} \pi + \sqrt{3}$ "

《参考》



斜線の部分の面積を S とすると

$$S = \triangle OPH + \text{扇形 } OPQ$$

$$= \frac{1}{2} \cdot 1 \cdot 2 \sin \frac{\pi}{3} + \frac{1}{2} \cdot 2^2 \cdot \frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

$$\int_{-1}^1 \sqrt{4-x^2} dx = 2 \int_0^1 \sqrt{2^2-x^2} dx$$

$$= 2S = 2 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) = \sqrt{3} + \frac{2}{3}\pi$$

3. 極限值 $\lim_{n \rightarrow \infty} \frac{1}{n^7} \sum_{k=1}^n k^6$ の値を求めよ。

(解) 区間 $[0, 1]$ を n 等分して n 個の小区分に付けて

$$x_k = \frac{1-0}{n} \cdot k = \frac{k}{n}, \quad \Delta x_k = \frac{1-0}{n} = \frac{1}{n}$$

($n = 1, 2, 3, \dots, n$) とおくと、

定積分の定義によって

$$\lim_{n \rightarrow \infty} \frac{1}{n^7} \sum_{k=1}^n k^6 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k}{n} \right)^6 \cdot \frac{1}{n}$$

$$= \lim_{\Delta x_k \rightarrow 0} \sum_{k=1}^n x_k^6 \Delta x_k = \int_0^1 x^6 dx$$

$$= \left[\frac{1}{7} x^7 \right]_0^1 = \frac{1}{7} [x^7]_0^1 = \frac{1}{7} (1^7 - 0^7) = \frac{1}{7} \quad "$$

4. $\int_1^x f(t) dt = x^2 - ax + 2$ を満たす関数 $f(x)$

と定数 a を求めよ。

$$(解) \frac{d}{dx} \int_1^x f(t) dt = \frac{d}{dx} (x^2 - ax + 2)$$

$$f(x) = 2x - a \quad f(t) = 2t - a$$

$$\int_1^x f(t) dt = \int_1^x (2t - a) dt = [t^2 - at]_1^x$$

$$= (x^2 - ax) - (1^2 - a \cdot 1) = x^2 - ax + a - 1$$

これと与式を比べて $a - 1 = 2$

$$a = 3 \quad " \quad , \quad f(x) = 2x - 3 \quad "$$